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Abstract

This report contains an account of geostrophic vortices on a rotating sphere. A vortex of this type is characterized by a singular spherical harmonic of degree ν and order zero. This function has a role analogous to that of the Bessel function which characterizes a rectilinear geostrophic vortex in a rotating plane.

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1. Introduction

A thin spherical layer of an incompressible, inviscid fluid which is held on the surface of a rotating ball by gravitation can be taken for some purposes as an approximation to the Earth's atmosphere. The analysis of the two dimensional vortical motion in such a layer should be useful for the understanding of certain observed meteorological phenomena.

In a previous report a linearized analysis was presented under the assumption that the outer surface of the layer is a sphere always concentric to the ball representing the Earth. This report presents a linear analysis in which the outer surface of the fluid layer is allowed to be free but the Coriolis force is assumed to be independent of latitude. This assumption defines a motion which can be regarded as a first approximation to the actual motion even if the fluid layer is not confined to a narrow band bounded by two circles of latitude.

The investigation covers the case in which tangential accelerations are neglected in comparison with the Coriolis force. In other words, we study geostrophic vortices on a sphere as contrasted with geostrophic vortices on a plane. The latter have been discussed by several authors in connection with a tangential plane approximation to the motion of a thin layer of fluid covering a rotating ball. References can be found in Morikawa, [1].

2. Equations of Motion

Let ρ denote the distance of a point from the center of a ball E of large radius a which rotates with constant angular velocity ω about a polar axis. Let ϕ and θ denote respectively the longitude and the colatitude of a point on the rotating spherical surface S of E. Let $\rho = a$ and $\rho = a + h(\phi, \theta, t)$ represent two surfaces which contain an incompressible, inviscid fluid which is gravitationally attracted by E. Suppose h is small compared with a.

The general problem is to find the motion of the fluid after its constant rotatory motion is disturbed by the sudden creation, at a reference time $t = 0$, of concentrated vortices normal to S.

The velocity of a fluid particle relative to S is defined by the components

$$u = (\rho \sin \theta) \frac{d\phi}{dt} = \text{tangential component toward the east;}$$

$$v = - \rho \frac{d\theta}{dt} = \text{tangential component toward the north;}$$

$$w = \frac{d\rho}{dt} = \text{radial component.}$$

If the only body force acting is that due to the gravitational potential G of E, then the basic hydrodynamical equations which define the motion of the fluid relative to S are the continuity equation

$$\rho \frac{\partial w}{\partial \rho} + 2w + \frac{1}{\sin \theta} \left[\frac{\partial u}{\partial \phi} - \frac{\partial (v \sin \theta)}{\partial \theta} \right] = 0$$

and the momentum equations

$$\frac{du}{dt} - \frac{uv \cot\theta}{\rho} + \frac{uw}{\rho} + 2\omega w \sin\theta - 2\omega v \cos\theta = -\frac{1}{\delta_o \rho \sin\theta}\frac{\partial p_1}{\partial \phi} ;$$

$$\frac{dv}{dt} + \frac{vw}{\rho} + \frac{u^2 \cot\theta}{\rho} + 2\omega u \cos\theta = \frac{1}{\delta_o \rho}\frac{\partial p_1}{\partial \theta} ;$$

$$\frac{dw}{dt} - \frac{v^2}{\rho} - \frac{u^2}{\rho} - 2\omega u \sin\theta = -\frac{1}{\delta_o}\frac{\partial p_1}{\partial \rho} .$$

In these equations the differential operator with respect to the time means

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{u}{\rho \sin\theta}\frac{\partial}{\partial\phi} - \frac{v}{\rho}\frac{\partial}{\partial\theta} + w\frac{\partial}{\partial\rho} .$$

The symbol δ_o denotes the constant density of the fluid; and p stands for the pressure in

$$p_1 = p + \delta_o G - \frac{\delta_o \rho^2 \omega^2 \sin^2\theta}{2}$$

which can be referred to as the modified pressure.

Since h is supposed to be small compared with the large radius a of E; and since

$$w(\phi, \theta, a, t) = 0$$

let us neglect the radial velocity and the radial variation of u and v. Let us assume that $G = g\rho$ where g is a constant; and that the centrifugal effects manifested by the partial derivatives of $\frac{\delta_o \rho^2 \omega^2 \sin^2\theta}{2}$ can be ignored. Let us also assume that the motion is such that the nonlinear terms in the tangential momentum equations can be neglected; and that the radial momentum equation can be replaced with the hydrostatic law

$$p(\phi, \theta, \rho, t) = g\delta_0(h + a - \rho)$$

which satisfies the condition that the pressure is zero at the free surface $\rho = a+h$.

Under the above assumptions and with the notation

$$\tilde{u}(\phi, \theta, t) = u(\phi, \theta, a, t) \ ,$$

$$\tilde{v}(\phi, \theta, t) = v(\phi, \theta, a, t) \ ;$$

an approximation to the motion is determined by the equations

$$\frac{\partial w}{\partial \rho} + \frac{1}{a \sin \theta}\left[\frac{\partial \tilde{u}}{\partial \phi} - \frac{\partial(\tilde{v} \sin \theta)}{\partial \theta}\right] = 0 \ ; \tag{2.1}$$

$$\frac{\partial \tilde{u}}{\partial t} - 2\omega\tilde{v} \cos \theta = -\frac{g}{a \sin \theta}\frac{\partial h}{\partial \phi} \ ; \tag{2.2}$$

$$\frac{\partial \tilde{v}}{\partial t} + 2\omega\tilde{u} \cos \theta = \frac{g}{a}\frac{\partial h}{\partial \theta} \ . \tag{2.3}$$

An integration of (2.1) gives

$$w(\phi, \theta, a+h, t) = \frac{h}{a \sin \theta}\left[\frac{\partial(\tilde{v} \sin \theta)}{\partial \theta} - \frac{\partial \tilde{u}}{\partial \phi}\right] .$$

The kinematic condition at the free surface is

$$w(\phi, \theta, a+h, t) = \frac{dh}{dt}$$

and hence

$$\frac{dh}{dt} = \frac{h}{a \sin \theta} \left[\frac{\partial(\tilde{v} \sin \theta)}{\partial \theta} - \frac{\partial \tilde{u}}{\partial \phi} \right] . \tag{2.4}$$

If $h(\phi\theta, 0) = h_o = \text{const.}$; and if we introduce

$$\eta(\phi, \theta, t) = \frac{h - h_o}{h_o} ,$$

assuming that η is small almost everywhere, then a linearization of (2.4) yields

$$\eta_t = \frac{1}{a \sin \theta} \left[\frac{\partial(\tilde{v} \sin \theta)}{\partial \theta} - \frac{\partial \tilde{u}}{\partial \phi} \right] \tag{2.5}$$

while the momentum equations become

$$\frac{\partial \tilde{u}}{\partial t} - 2\omega \tilde{v} \cos \theta = - \frac{g h_o}{a \sin \theta} \frac{\partial \eta}{\partial \phi} ; \tag{2.6}$$

$$\frac{\partial \tilde{v}}{\partial t} + 2\omega \tilde{u} \cos \theta = \frac{g h_o}{a} \frac{\partial \eta}{\partial \theta} . \tag{2.7}$$

3. Geostrophic Vortices

If we neglect the variation of the Coriolis force with latitude and take

$$\omega \cos \theta = \omega \cos \theta_1 = \omega_1$$

the last equations of Section 2 reduce to

$$\eta_t = \frac{1}{a \sin \theta} \left[\frac{\partial(\tilde{v} \sin \theta)}{\partial \theta} - \frac{\partial \tilde{u}}{\partial \phi} \right] ; \tag{3.1}$$

$$(3.2) \qquad \frac{\partial \tilde{u}}{\partial t} - 2\omega_1 \tilde{v} = - \frac{gh_o}{a \sin \theta} \frac{\partial \eta}{\partial \phi} ;$$

$$(3.3) \qquad \frac{\partial \tilde{v}}{\partial t} + 2\omega_1 \tilde{u} = \frac{gh_o}{a} \frac{\partial \eta}{\partial \theta} .$$

Hereafter we confine ourselves to a study of these equations. As will be explained in the sequel, they lead to what are called geostrophic vortices.

The elimination of η from (3.2) and (3.3) leads to

$$(3.4) \qquad \frac{\partial}{\partial t} \left\{ \frac{1}{a \sin \theta} \left[\frac{\partial u \sin \theta}{\partial \theta} + \frac{\partial v}{\partial \phi} \right] \right\} = 2\omega_1 \eta_t .$$

The quantity

$$\zeta = \frac{1}{a \sin \theta} \left[\frac{\partial (\tilde{u} \sin \theta)}{\partial \theta} + \frac{\partial \tilde{v}}{\partial \phi} \right]$$

is the radial component of vorticity; and by integration of (3.4) we have

$$(3.5) \qquad \frac{1}{a \sin \theta} \left[\frac{\partial (\tilde{u} \sin \theta)}{\partial \theta} + \frac{\partial \tilde{v}}{\partial \phi} \right] = 2\omega_1 \eta + \zeta(\phi, \theta, 0) .$$

This implies that we can use (3.5), (3.2) and (3.3) as a basic set of equations instead of (3.1); (3.2) and (3.3).

Equations (3.2) and (3.3) give

$$(3.6) \qquad \frac{\partial^2 \tilde{u}}{\partial t^2} + 4\omega_1^2 \tilde{u} = \frac{gh_o}{a} \left[2\omega_1 \frac{\partial \eta}{\partial \theta} - \frac{1}{\sin \theta} \frac{\partial^2 \eta}{\partial t \partial \phi} \right]$$

and

$$(3.7) \qquad \frac{\partial^2 \tilde{v}}{\partial t^2} + 4\omega_1^2 \tilde{v} = \frac{gh_o}{a}\left[\frac{2\omega_1}{\sin\theta}\frac{\partial\eta}{\partial\phi} + \frac{\partial^2\eta}{\partial t\partial\theta}\right].$$

If these are used to eliminate $\tilde{u}$ and $\tilde{v}$ from (3.5), we find

$$(3.8) \qquad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial\eta}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\eta}{\partial\phi^2}$$

$$- \frac{a^2}{gh_o}\left[4\omega_1^2\eta + \frac{\partial^2\eta}{\partial t^2}\right] = \frac{a^2 2\omega_1 \zeta(\phi,\theta,0)}{gh_o}.$$

For a vortex of strength μ concentrated at (ϕ_1, θ_1) when $t = 0$ we take

$$\frac{a^2 2\omega_1 \zeta(\phi,\theta,0)}{gh_o} = \frac{a^2 2\omega_1}{gh_o}\cdot\mu\cdot\frac{\delta(\phi - \phi_1)\delta(\theta - \theta_1)}{a^2 \sin\theta_1}$$

where δ symbolizes the Dirac delta function.

The theory of Laplace transforms can be used to show that the steady state solution of (3.8) corresponding to a concentrated vortex is such that

$$(3.9) \qquad L_{t\to\infty}\,\eta(\phi,\theta,t) = \frac{2\omega_1\psi(\phi,\theta)}{gh_o}$$

where ψ must satisfy

$$(3.10) \qquad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} - \frac{4a^2\omega_1^2\psi}{gh_o}$$

$$= \frac{\mu\delta(\phi - \phi_1)\delta(\theta - \theta_1)}{\sin\theta_1}.$$

The function ψ does not depend on t and the associated time independent velocity comonents are

$$(3.11) \qquad U = L_{t\to\infty}\tilde{u}(\phi,\theta,t) = \frac{1}{a}\frac{\partial\psi}{\partial\theta} \ ;$$

$$(3.12) \qquad V = L_{t\to\infty}\tilde{v}(\phi,\theta,t) = \frac{1}{a \sin\theta}\frac{\partial\psi}{\partial\phi} \ .$$

Note that (3.10) is a consequence of the equations (3.11), (3.12) and the vorticity equation

$$(3.13) \quad \frac{1}{a \sin\theta}\left[\frac{\partial}{\partial\theta}(U \sin\theta) + \frac{\partial V}{\partial\phi}\right] = 2\omega_1 L_{t\to\infty}\eta + \zeta(\phi,\theta,0)$$

$$= \frac{4\omega_1^2\psi}{gh_o} + \frac{\mu\delta(\phi - \phi_1)\delta(\theta - \theta_1)}{a^2\sin\theta_1} \ .$$

In other words, (3.10) is implied by (3.5); (3.2) and (3.3) when we ignore inertial forces.

The motion in a thin planar layer of fluid tangential to the surface of the Earth is often used as an approximation to the actual motion of the Earth's atmosphere. For such an approximation the analogues of (3.11); (3.12) and (3.13) are

$$(3.14) \qquad U = \frac{\partial\chi}{\partial r} \ ;$$

$$(3.15) \qquad V = \frac{1}{r}\frac{\partial\chi}{\partial\phi} \ ;$$

and

$$(3.16) \qquad \frac{1}{r}\left[\frac{\partial rU}{\partial r} + \frac{\partial V}{\partial\phi}\right] = \frac{4\omega_1^2\chi}{gh_o} + \frac{\mu\delta(\phi - \phi_1)\delta(r - r_1)}{r_1} \ .$$

In terms of the polar coordinates (r, ϕ), these show that χ must satisfy

$$(3.17) \qquad \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \chi}{\partial \phi^2} - \frac{4\omega_1^2 \chi}{gh_o} = \frac{\mu \delta(\phi - \phi_1)\delta(r - r_1)}{r_1} .$$

The only physically admissible solution of (3.17) is

$$(3.18) \qquad \chi = -\frac{\mu}{2\pi} K_o \left[\frac{2\omega_1}{\sqrt{gh_o}} \sqrt{r^2 + r_1^2 - 2rr_1 \cos(\phi - \phi_1)} \right]$$

where $K_o(\ )$ denotes the zeroth order modified Bessel function of the second kind. This defines what is called a geostrophic vortex. The motion of various configurations of such vortices has been studied by various authors, notably G. K. Morikawa [1], [2] whose papers contain detailed explanations and other references. In keeping with what seems to be accepted terminology we can say that (3.10) defines a geostrophic vortex on a sphere.

Our object now is to investigate the nature of the solution of (3.10). This equation can be written as

$$(3.19) \qquad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \nu(\nu+1)\psi$$

$$= \frac{\mu\delta(\phi - \phi_1)\delta(\theta - \theta_1)}{\sin\theta_1}$$

where

$$(3.20) \qquad \nu(\nu+1) = -\frac{4\omega_1^2 a^2}{gh_o} .$$

The solution of (3.19) subject to

$$\alpha < \theta < \beta \; ; \quad 0 \leq \phi \leq 2\pi \; ;$$

and prescribed boundary conditions on the circles of latitude

$$\theta = \alpha \; ; \quad \theta = \beta \; ;$$

namely a Green's function for a zone, can be expressed in the form

$$(3.21) \qquad \psi = \frac{\mu}{2\pi} \sum_{m=0}^{\infty} [A_m P_\nu^m(\cos\theta) + B_m Q_\nu^m(\cos\theta)] \cos m(\phi - \phi_1)$$

where P_ν^m and Q_ν^m are solutions of

$$\frac{d}{dz}(1 - z^2)\frac{df}{dz} + \left[\nu(\nu+1) - \frac{m^2}{1 - z^2}\right] f = 0 \; .$$

The appropriate series (3.21) can be summed for the case in which the zone becomes a sphere. However, since the details of the analysis based on the use of (3.21) are lengthy, let us proceed in a different way.

The general solution of

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial\psi}{\partial\theta} + \nu(\nu+1)\psi = 0$$

can be expressed in the form

$$\psi = c_1 P_\nu(-\cos\theta) + c_2 P_\nu(\cos\theta)$$

when ν is not an integer as is the case when ν is defined by

$$\nu(\nu+1) = -\frac{4\omega_1^2 a^2}{g h_o}$$

or

$$\nu = -\frac{1}{2} + i\lambda$$

where

$$\lambda = \frac{1}{2}\sqrt{\frac{16\omega_1^2 a^2}{gh_o} - 1}\ .$$

In the neighborhood of the north pole $P_\nu(\cos\theta)$ is continuous and $P_\nu(1) = 1$; but

$$\begin{aligned} P_\nu(-\cos\theta) &= P_{-\frac{1}{2}+i\lambda}(-\cos\theta) \\ &= \frac{2}{\pi}\cosh\lambda\pi \int_0^\infty \frac{\cos\lambda x dx}{\sqrt{2(\cosh x - \cos\theta)}} \end{aligned}$$

behaves like

$$P_\nu(-\cos\theta) \sim \frac{2\sin\nu\pi}{\pi}\cdot \ln\theta\ .$$

This suggests that if a vortex is concentrated at (ϕ_1, θ_1); if d_1 is the geodesic distance from this point to an arbitrary point (ϕ, θ) on S; and if γ_1 is the angle $\gamma_1 = d_1/a$, then

$$\frac{\mu}{4\sin\nu\pi} P_\nu(-\cos\gamma_1) \equiv \frac{\mu}{4\sin\nu\pi} P_\nu\left[\begin{aligned} &-\cos\theta\cos\theta_1 \\ &-\sin\theta\sin\theta_1\cos(\phi-\phi_1)\end{aligned}\right]$$

is the fundamental solution of (3.19).

It can now be verified by direct substitution that

$$\psi = \frac{\mu}{4\sin\nu\pi} P_\nu(-\cos\gamma_1) \tag{3.22}$$

does indeed satisfy (3.19). Furthermore, computations with the velocity components

$$U = \frac{\mu}{4a \sin \nu\pi} \frac{\partial}{\partial\theta} P_\nu(-\cos \gamma_1) ;$$

$$V = \frac{\mu}{4a \sin \theta \sin \nu\pi} \frac{\partial}{\partial\phi} P_\nu(-\cos \gamma_1) ;$$

show that in the neighborhood of (ϕ_1, θ_1), (3.22) defines a vortical motion; but the vortex point itself remains at rest — it possesses no autonomous motion. In fact, it can be shown that

$$L_{a \to \infty} \frac{\mu}{4 \sin \nu\pi} P_\nu(-\cos \gamma_1)$$

$$= - \frac{\mu}{4\pi} K_o \left[\frac{2\omega_1}{\sqrt{gh_o}} \sqrt{r^2 + r_1^2 - 2rr_1 \cos (\phi - \phi_1)} \right] .$$

Hence we can say that (3.22) represents a geostrophic vortex on a rotating sphere in the same way as we say that

$$\chi = - \frac{\mu}{4\pi} K_o \left[\frac{2\omega_1}{\sqrt{gh_o}} \sqrt{r^2 + r_1^2 - 2rr_1 \cos (\phi - \phi_1)} \right]$$

represents a geostrophic vortex on a rotating plane.

For an arbitrary distribution of n vortices on the sphere the function ψ, which can be regarded as a stream function, is

$$\psi = \frac{1}{4 \sin \nu\pi} \sum_{i=1}^{n} \mu_i P_\nu(-\cos \gamma_i)$$

where $\gamma_i = d_i/a$ and d_i is the geodesic distance on S from the point of concentration of the i-th vortex, (ϕ_i, θ_i), to an arbitrary point (ϕ, θ). That is,

$$\cos \gamma_i = \cos \theta \cos \theta_i + \sin \theta \sin \theta_i \cos (\phi - \phi_i) .$$

The associated velocity field is given by

$$U = \frac{1}{a}\frac{\partial\psi}{\partial\theta} = \frac{1}{4a \sin \nu\pi}\frac{\partial}{\partial\theta}\sum_{i=1}^{n}\mu_i P_\nu(-\cos \gamma_i) ;$$

$$V = \frac{1}{a \sin \theta}\frac{\partial\psi}{\partial\phi} = \frac{1}{4a \sin \theta \sin \nu\pi}\frac{\partial}{\partial\phi}\sum_{i=1}^{n}\mu_i P_\nu(-\cos \gamma_i) .$$

Since

$$U = a \sin \theta \frac{d\phi}{dt} ; \quad V = -a \frac{d\theta}{dt}$$

the equations of motion of the vortex at (ϕ_k, θ_k) are

$$\frac{d\phi_k}{dt} = \frac{1}{4a^2 \sin \theta \sin \nu\pi}\left|\frac{\partial}{\partial\theta}\sum_{\substack{i=1\\ i\neq k}}^{n}\mu_i P_\nu(-\cos \gamma_i)\right|_{\substack{\phi=\phi_k\\ \theta=\theta_k}}$$

and

$$\frac{d\theta_k}{dt} = -\frac{1}{4a^2 \sin \theta \sin \nu\pi}\left|\frac{\partial}{\partial\phi}\sum_{\substack{i=1\\ i\neq k}}^{n}\mu_i P_\nu(-\cos \gamma_i)\right|_{\substack{\phi=\phi_k\\ \theta=\theta_k}} .$$

We are now in a position to develop a general theory for geostrophic vortices on a sphere analogous to the basic Helmholtz-Kirchoff theory for rectilinear vortices normal to a plane. Instead of doing this here it seems more useful to examine a few special cases which may be approximately applicable to certain physical situations.

4. Geostrophic Vortices in the Northern Hemisphere. Either Velocity Component Zero at the Equator

Let us turn now to the case of a concentrated geostrophic vortex in the northern hemisphere subject to the condition that the normal velocity of the fluid at the equator is zero. For this case the stream function ψ must satisfy

$$(4.1) \qquad \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial\psi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \nu(\nu+1)\psi = \frac{\mu\delta(\phi-\phi_1)\delta(\theta-\theta_1)}{\sin\theta_1}$$

for

$$0 \leq \theta, \theta_1 < \frac{\pi}{2}; \quad 0 \leq \phi, \phi_1 \leq 2\pi .$$

The boundary condition is

$$(4.2) \qquad V(\phi, \tfrac{\pi}{2}) = \frac{1}{a}\psi_\phi(\phi, \tfrac{\pi}{2}) = 0$$

or

$$\frac{2\omega_1}{gh_0}\psi(\phi, \tfrac{\pi}{2}) = L_{t\to\infty}\eta(\phi,\theta,t) = \text{const.}$$

We have supposed that $\eta(\phi, \frac{\pi}{2}, 0) = 0$; and we now suppose that $\eta(\phi, \frac{\pi}{2}, t) = 0$ for all t. The boundary condition (4.2) then becomes

$$(4.3) \qquad \psi(\phi, \tfrac{\pi}{2}) = 0 .$$

The solution of (4.1) which satisfies (4.3) can be formed from the fundamental solution

$$\frac{\mu}{4 \sin \nu\pi} P_\nu(-\cos \gamma_1)$$

is we assume that the idea of reflection across the equator will lead toward the solution. In fact, it is easy to verify that

$$(4.4) \qquad \psi = \frac{\mu}{4 \sin \nu\pi} \left\{ \begin{aligned} & P_\nu[-\cos\theta \cos\theta_1 - \sin\theta \sin\theta_1 \cos(\phi - \phi_1)] \\ & -P_\nu[\cos\theta \cos\theta_1 - \sin\theta \sin\theta_1 \cos(\phi - \phi_1)] \end{aligned} \right\}$$

is the desired solution. With respect to the whole sphere, ψ of (4.4) has concentrated vortices at (ϕ_1, θ_1) and $(\phi_1, \pi - \theta_1)$. It should be noted that if we seek a stream function for a geostrophic vortex on a rotating plane such that the normal velocity of the fluid is zero along a circle whose center is at the center of rotation, then a result analogous to (4.4) does not exist.

The velocity components of the motion defined by (4.4) are

$$U = \frac{1}{a} \frac{\partial\psi}{\partial\theta}$$

$$= \frac{\mu}{4a \sin \nu\pi} \left\{ \begin{aligned} & P'_\nu \begin{bmatrix} -\cos\theta \cos\theta_1 \\ -\sin\theta \sin\theta_1 \cos(\phi - \phi_1) \end{bmatrix} \cdot \begin{bmatrix} \sin\theta \cos\theta_1 \\ -\cos\theta \sin\theta_1 \cos(\phi - \phi_1) \end{bmatrix} \\ & +P'_\nu \begin{bmatrix} \cos\theta \cos\theta_1 \\ -\sin\theta \sin\theta_1 \cos(\phi - \phi_1) \end{bmatrix} \cdot \begin{bmatrix} \sin\theta \cos\theta_1 \\ +\cos\theta \sin\theta_1 \cos(\phi - \phi_1) \end{bmatrix} \end{aligned} \right\}$$

and

$$V = \frac{1}{a \sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$= \frac{\mu}{4a \sin\nu\pi}\left\{\begin{array}{l} P_\nu'\begin{bmatrix} -\cos\theta\cos\theta_1 \\ -\sin\theta\sin\theta_1\cos(\phi-\phi_1)\end{bmatrix}\cdot\sin\theta_1\sin(\phi-\phi_1) \\ -P_\nu'\begin{bmatrix} \cos\theta\cos\theta_1 \\ -\sin\theta\sin\theta_1\cos(\phi-\phi_1)\end{bmatrix}\cdot\sin\theta_1\sin(\phi-\phi_1)\end{array}\right\}$$

The equations for the motion of the vortex concentrated at (ϕ_1,θ_1) come from evaluating at (ϕ_1,θ_1) the velocity field due to the vortex at $(\phi_1,\pi-\theta_1)$. They are

$$\dot{\phi}_1 = \frac{d\phi_1}{dt} = \frac{\cos\theta_1}{2a^2 \sin\nu\pi} P_\nu'(\cos 2\theta_1) ;$$

$$\dot{\theta}_1 = \frac{d\theta_1}{dt} = 0 .$$

From these we see that the vortex remains on its initial circle of latitude which it traverses with constant angular velocity.

The function ψ corresponding to a concentrated vortex in the northern hemisphere with the boundary condition

$$U(\phi,\tfrac{\pi}{2}) = 0$$

can also be found by appealing to a reflection process. It is

$$\psi = \frac{\mu}{4\sin\nu\pi}\left\{\begin{array}{l} P_\nu[-\cos\theta\cos\theta_1 - \sin\theta\sin\theta_1\cos(\phi-\phi_1)] \\ +P_\nu[\cos\theta\cos\theta_1 - \sin\theta\sin\theta_1\cos(\phi-\phi_1)]\end{array}\right\} .$$

It should be remarked, however, that a reflection technique for the determination of solutions of (3.19) for a general zone is unknown.

If the northern component of the velocity of the fluid is zero at the equator; and if we have a polar vortex of strength μ_o with three others of equal strength μ which at t = 0 are situated at

$$(\phi_1, \theta_1)\ ; \quad (\phi + \frac{2\pi}{3}, \theta_1)\ ; \quad (\phi_1 + \frac{4\pi}{3}, \theta_1)$$

in the northern hemisphere, then the stream function is

$$\psi = \frac{1}{4 \sin \nu\pi} \left[\begin{array}{c} \mu_o[P_\nu(-\cos\theta) - P_\nu(\cos\theta)] \\ \\ +\mu \displaystyle\sum_{n=0}^{2} \left\{ \begin{array}{l} P_\nu[-\cos\theta\cos\theta_1 - \sin\theta\sin\theta_1 \cos(\phi - \phi_1 - \frac{n2\pi}{3}] \\ -P_\nu[\cos\theta\cos\theta_1 - \sin\theta\sin\theta_1\cos(\phi - \phi_1 - \frac{n2\pi}{3}] \end{array} \right\} \end{array} \right]$$

A calculation shows that the polar vortex remains stationary while the paths of the other three are determined by

$$\dot{\theta}_k = 0$$

$$\dot{\phi}_k = \frac{1}{4a^2 \sin \nu\pi} \left\{ \begin{array}{c} \mu_o[P'_\nu(-\cos\theta_1) + P'_\nu(\cos\theta_1)] \\ \\ +\mu \left[\begin{array}{l} 3\cos\theta_1 P'_\nu(\frac{3}{2}\sin^2\theta_1 - 1) \\ -2\cos\theta_1 P'_\nu(\cos 2\theta_1) \\ +\cos\theta_1 P'_\nu(1 - \frac{1}{2}\sin^2\theta_1) \end{array} \right] \end{array} \right\} \tag{4.5}$$

$$= \Omega\ .$$

In other words, the position of the vortices subsequent to $t = 0$ are given by

$$(\phi_0, 0) \ ;$$

$$(\phi_1 + \Omega t, \theta_1) \ ; \quad (\phi_1 + \frac{2\pi}{3} + \Omega t, \theta_1) \ ; \quad (\phi_1 + \frac{4\pi}{3} + \Omega t, \theta_1) \ .$$

Equation (4.5) shows that for a certain strength μ_0 the angular velocity Ω becomes zero and all the vortices remain stationary. The stability of this equilibrium configuration can be investigated by using the methods explained in the paper by Morikawa [1]; and the paper by Morikawa and Swenson [2].

References

[1] Morikawa, G. K., Geostrophic Vortex Motion, Journal of Meteorology, Vol. 17, No. 2, April 1960, pp. 148-158.

[2] Morikawa, G. K., and Swenson, E. V., Interacting Motion of Rectilinear Geostrophic Vortices, The Physics of Fluids, Vol. 14, No. 6, June 1971, pp. 1058-1073.